

C^* –non–linear second quantization

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Abstract

We construct an inductive system of C^* –algebras each of which is isomorphic to a finite tensor product of copies of the one mode n –th degree polynomial extension of the usual Weyl algebra constructed in our previous paper [10].

We prove that the inductive limit C^* –algebra is factorizable and has a natural localization given by a family of C^* –sub–algebras each of which is localized on a bounded Borel subset of \mathbb{R} . Finally we prove that the corresponding families of Fock states, defined on the inductive family of C^* –algebras, is projective if and only if $n = 1$. This is a weak form of the no–go theorems which emerge in the study of representations of current algebras over Lie algebras.

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1 Introduction: the C^* -non-linear quantization program

The present paper is a contribution to the program of constructing a theory of renormalized higher powers of quantum white noise (RPWN) or equivalently of non relativistic free Boson fields.

This program has an old history, but the approach discussed here started in 1999 with the construction of the Fock representation for the renormalized square of Boson white noise [2]. This result motivated a large number of papers extending it in different directions and exhibiting connections with almost all fields of mathematics, see for example [18] for the case of free white noise, [1] for the connection with infinite divisibility and for the identification of the vacuum distributions of the generalized fields with the three non standard Meixner classes, [3] and [17] for finite temperature representations, [9] for the construction of the Fock functor, the survey [5] and the paper [6] for the connections with conformal field theory and with the Virasoro–Zamolodchikov hierarchy, [7] for the connections between renormalization and central extensions.

The problem is the following. One starts with the Schroedinger representation of the Heisenberg real $*$ -Lie algebra with skew-adjoint generators iq (imaginary unit times position), $-ip$ ($-i$ times momentum), $E := i1$ (i times central element) and relations $[iq, -ip] = i1$.

The universal enveloping algebra of this Lie algebra, called for brevity the *full oscillator algebra* (FOA), can be identified with the algebra of differential operators in one real variable with complex polynomial coefficients.

The continuous analogue of the Heisenberg Lie algebra is the non relativistic

free boson field algebra, also called the current algebra over \mathbb{R} of the Heisenberg algebra, whose only non zero commutation relations are, in the sense of operator valued distributions:

$$[q_s, p_t] = \delta(s - t)1 \quad ; \quad s, t \in \mathbb{R}$$

The notion of current algebra has been generalized from the Heisenberg algebra to more general \ast -Lie algebras (see Araki's paper [12] for a mathematical treatment and additional references): in this case the self-adjoint generators of the Cartan sub-algebras are called generalized fields.

Notice that the definition of current algebra of a given Lie algebra is independent of any representation of this algebra, i.e. it does not require to fix a priori a class of states on this algebra.

One can speak of *\ast -Lie algebra second quantization* to denote the transition from the construction of unitary representations of a \ast -Lie algebra to the construction of unitary representations of its current algebra over a measurable space (typically \mathbb{R} with its Borel structure).

Contrarily to the discrete case, the universal enveloping algebra of the current algebra over \mathbb{R} of the Heisenberg algebra is ill defined because of the emergence of higher powers of the δ -function. This is the mathematical counterpart of the old problem of defining powers of local quantum fields.

Any rule that, giving a meaning to these powers, defines a \ast -Lie algebra structure, is called a renormalization procedure. The survey [5] describes two inequivalent renormalization procedures and the more recent paper [7] shows the connection between them.

The second step of the program, after renormalization, is the construction of unitary representations of the resulting \ast -Lie algebra. This step, which is the most difficult one because of the no-go theorems (see discussion below), is usually done by fixing a state and considering the associated cyclic representation. At the moment, even in the first order case, i.e. for usual fields, the explicitly constructed representations are not many, they are essentially reduced to gaussian (quasi-free) representations. Moreover any gaussian representation can be obtained, by means of a standard construction, from the Fock representation which is characterized by the property that the cyclic vector, called vacuum, is in the kernel of the annihilation operators.

This property has been taken as an heuristic principle to define the notion of *Fock state* also in the higher order situations (see [5] for a precise definition). It can be proved that, for all renormalization procedures considered up to now, the Fock representation and the Fock state are factorizable, in the sense

of Araki–Woods [11]. This property poses an obstruction to the existence of such representation, namely that the restriction of the Fock state on any factorizable Cartan sub-algebra must give rise to a classical infinitely divisible process. If this is not the case then *no Fock representation, and more generally no cyclic representation associated to a factorizable state, can exist. When this is the case we say that a no-go theorem holds.* Nowadays several instances of no-go theorems are available. The simplest, and probably most illuminating one, concerns the Schroedinger algebra, which is the Lie algebra generated by the powers ≤ 2 of p and q (see [18], also [1] and [8] for stronger results). This result implies that there is *no natural analogue of the Fock representation for the current algebra over \mathbb{R} (for any $d \in \mathbb{N}$) of the FOA.* On the other side we know (see [1], [2] and the above discussion) that *for some sub-algebras of current algebras of the FOA such a representation exist.* This naturally rises the problem to characterize these sub-algebras.

Since a full characterization at the moment is not available, a natural intermediate step towards such a characterization is to produce nontrivial examples.

To this goal a family of natural candidates is provided by the \ast -Lie-sub-algebras of the FOA consisting of the real linear combinations of the derivation operator and the polynomials of degree less or equal than a fixed natural integer n . Thus the generic element of such an algebra has the form

$$up + P(q) \quad ; \quad u \in \mathbb{R}$$

where P is a polynomial of degree n with real coefficients. For $n = 1$ one finds the Heisenberg algebra; for $n = 2$ the Galilei algebra and, for $n > 2$, some nilpotent Lie algebras well studied in mathematics [13], [14] [15], [16] but up to now, with the notable exception of the Galilei algebra ($n = 2$), not considered in physics.

These \ast -Lie-algebras enjoy two very special properties:

- (i) no renormalization is required in the definition of the associated current algebra over \mathbb{R} ;
- (ii) in the Schroedinger representation of the FOA the skew-adjoint elements of these sub-algebras can be explicitly exponentiated giving rise to a nonlinear generalization of the Weyl relations and of the corresponding Heisenberg group. This was done in the paper [10].

Property (i) supports the hope of the existence of the Fock representation for the above mentioned current algebra. A direct proof of this fact could be

obtained by proving the infinite divisibility of all the vacuum characteristic functions of the generalized fields. Unfortunately even in the case $n = 2$, in which this function can be explicitly calculated, a direct proof of infinite divisibility can be obtained only for a subset of the parameters which define the generalized fields, but not for all, and this problem is challenging the experts of infinite divisibility since several years.

In the present paper we exploit property (ii) and the following heuristic considerations are aimed at making a bridge between the mathematical construction below and its potential physical interpretation.

Our goal is to construct a C^* -algebra whose generators can be *naturally* identified with the following formal expressions that we call the *non-linear Weyl operators*:

$$e^{i(p(f_0)+q^n(f_n)+\dots+q^1(f_1))} = e^{i\sum_{j\in\{0,1,\dots,n+1\}} L_j(f_j)} \quad (1)$$

The formal generators of the non-linear Weyl operators (called *non-linear fields*) are heuristically expressed as *powers of the standard quantum white noise (or free Boson field)*, i.e. the pair of operator valued distributions q_t, p_t with commutation relations

$$[q_s, p_t] = i\delta(t - s)$$

in the following way:

$$\begin{aligned} L_{n+1}(f_0) &:= p(f_0) = \int_{\mathbb{R}} f_0(t) p_t dt & ; & & L_0(f) &:= \int_{\mathbb{R}} f_k(t) q_t^0 dt := 1 \cdot \int f(s) ds \\ L_k(f) &:= q^k(f_k) = \int_{\mathbb{R}} f_k(t) q_t^k dt & ; & & k &\in \{0, 1, \dots, n\} \end{aligned} \quad (2)$$

When $n = 1$, the expressions (2) are well understood and various forms of second quantization are known. For example one can prove the unitarity of the Fock representation the exponentiability, inside it, of the generators (2) and the commutation relations satisfied by them. A different example, for $n = 1$, is provided by Weyl second quantization: in it, by heuristic calculations, one guesses the commutation relations that should be satisfied by any representation of the exponentials (1) and then one proves the existence of a C^* -algebra which realizes these commutation relations.

In the present paper we apply this approach to give a meaning to the exponentials (1) and in this sense we speak of C^* -*second quantization*.

To this goal we exploit the fact that, if π is a finite Borel partition of a

bounded Borel subset of \mathbb{R} , then there is a natural way to give a meaning to the generalized Weyl algebra with test functions constant on the sets of π . This is based on the identification of this algebra with the tensor product of $|\pi|$ (cardinality of π) *rescaled copies* of the one mode generalized Weyl algebra (see section 8). This identification strongly depends on the specific structure of the Lie algebra considered (see section 6 below).

Using this we construct an inductive system of C^* -algebras each of which is isomorphic to a finite tensor product of copies of the one mode generalized Weyl algebra **but the embeddings defining the inductive system are not the usual tensor product embeddings**.

The C^* -algebra, obtained as inductive limit from the above construction, is naturally interpreted as a C^* -*quantization*, over \mathbb{R} , of the initial $*$ -Lie algebra. This C^* -algebra has a *localization* given by a family of C^* -sub-algebras, each of which has a natural localization on bounded Borel subset of \mathbb{R} .

Moreover this system of local algebras is factorizable in the sense of Definition 8 below.

With this construction the problem of constructing unitary representations of the current algebra over \mathbb{R} of the initial $*$ -Lie algebra is reduced to the problem of finding representations of this C^* -algebra: the advantages of this transition from unbounded to bounded case are well known in the case of standard, first order, quantization.

In the last section of the paper it is shown that, although the Fock state is defined on each of the C^* -algebras of the inductive family, the corresponding family of states is projective if and only if $n = 1$ (i.e. for the usual Weyl algebra). This result can be considered as a C^* -version of the no-go theorems proved in [18], [1], [8] for different algebras.

The basic construction of the present paper can be extended to more general classes of $*$ -Lie algebras (for example the C^* -algebras associated to the renormalized square of white noise (RSWN)) and more general spaces (i.e. \mathbb{R}^d instead of \mathbb{R}).

2 The 1-mode n -th degree Heisenberg \ast -Lie algebra $heis_{\mathbb{R}}(1, n)$

Definition 1 For $n \in \mathbb{N}^*$ the 1-mode n -th degree Heisenberg algebra, denoted $heis_{\mathbb{R}}(1, n)$, is the pair

$$\{V_{n+2}, (L_j)_{j=0}^{n+1}\}$$

where:

- V_{n+2} is a $(n+2)$ -dimensional real \ast -Lie algebra;
- $(L_j)_{j=0}^{n+1}$ is a skew-adjoint linear basis of V_{n+2} ;
- the Lie brackets among the generators are given by

$$[L_i, L_j] = 0 \quad ; \quad \forall i, j \in \{0, 1, \dots, n\}$$

$$[L_{n+1}, L_k] = kL_{k-1} \quad ; \quad \forall k \in \{1, \dots, n\}, \quad L_{-1} := 0$$

Remark 1 1) Multiplying each of the generators $(L_j)_{j=0}^{n+1}$ by a strictly positive number, one obtains a new basis $(L'_j)_{j=0}^{n+1}$ of V_{n+2} satisfying the new commutation relations

$$[L'_{n+1}, L'_k] = kd_k L'_{k-1} \quad ; \quad \forall k \in \{0, \dots, n\}, \quad L'_{-1} := 0$$

In this case we speak of a *re-scaled copy* of the 1-mode n -th degree Heisenberg algebra.

- 2) Denoting $\mathbb{R}_n[X]$ the vector space of polynomials in one indeterminate with real coefficients and degree less or equal than n , the assignment of the basis $(L_j)_{j=0}^{n+1}$ uniquely defines the parametrization

$$\begin{aligned} (u, (a_k)_{k \in \{0, 1, \dots, n\}}) &\in \mathbb{R} \times \mathbb{R}_n[X] \equiv \mathbb{R}^{n+2} \mapsto \ell_0(u, P) := \\ &:= uL_{n+1} + \sum_{k \in \{0, 1, \dots, n\}} a_k L_k =: uL_{n+1} + P(L) \in heis_{\mathbb{R}}(1, n) \end{aligned} \quad (3)$$

of $heis_{\mathbb{R}}(1, n)$ by elements of $\mathbb{R}_n[X]$. When no confusion is possible we will use the identification

$$\ell_0(u, P) \equiv (u, (a_k)_{k \in \{0, 1, \dots, n\}}) \in \mathbb{R}^{n+2} \quad (4)$$

3 The Schroedinger representation and the polynomial Heisenberg group $\text{Heis}(1, n)$

Let $p, q, 1$ be the usual momentum, position and identity operators acting on the one mode boson Fock space

$$\mathcal{H}_1 = \Gamma(\mathbb{C}) = L^2(\mathbb{R}) \quad (5)$$

The maximal algebraic domain \mathcal{D}_{max} (see [4]), consisting of the linear combinations of vectors of the form

$$q^n p^k \psi_z \quad ; \quad k, n \in \mathbb{N}, z \in \mathbb{C}$$

where ψ_z is the exponential vector associated to $z \in \mathbb{C}$, is a dense subspace of $\Gamma(\mathbb{C})$ invariant under the action of p and q hence of all the polynomials in the two non commuting variables p and q . In particular, for each $n \in \mathbb{N}$, the real linear span of the set $\{i1, ip, iq, \dots, iq^n\}$, denoted $\text{heis}_{\mathbb{R}}(F, 1, n)$, leaves invariant the maximal algebraic domain \mathcal{D}_{max} . Hence the commutators of elements of this space are well defined on this domain and one easily verifies that they define a structure of $*$ -Lie algebra on $\text{heis}_{\mathbb{R}}(F, 1, n)$.

Lemma 1 In the above notations the map

$$L_{n+1} \mapsto ip \quad , \quad L_0 \mapsto i1 \quad , \quad L_k \mapsto iq^k \quad ; \quad k \in \{1, \dots, n\} \quad (6)$$

admits a unique linear extension from $\text{heis}_{\mathbb{R}}(1, n)$ onto $\text{heis}_{\mathbb{R}}(F, 1, n)$ which is a $*$ -Lie algebra isomorphism called *the Schroedinger representation* of the n -th degree Heisenberg algebra $\text{heis}_{\mathbb{R}}(1, n)$.

Proof. The linear space isomorphism property follows from the linear independence of the set $\{1, p, q, \dots, q^n\}$. The $*$ -Lie algebra isomorphism property follows from direct computation. \square

In [10] (Theorem 1) it is proved that the unitary operators

$$W(u, P) := e^{i(up + P(q))} \in \text{Un}(L^2(\mathbb{R})) \quad ; \quad (u, P) \in \mathbb{R} \times \mathbb{R}_n[X] \quad (7)$$

satisfy the following polynomial extension of the Weyl relations:

$$W(u, P)W(v, Q) = W((u, P) \circ (v, Q)) \quad ; \quad \forall (u, P), (v, Q) \in \mathbb{R} \times \mathbb{R}_n[X] \quad (8)$$

where

$$(u, P) \circ (v, Q) := (u + v, T_{u+v}^{-1}(T_u P + T_v S_u Q)) \quad (9)$$

and for any $u, w \in \mathbb{R}$, the linear operators $T_w, S_u : \mathbb{R}_n[X] \rightarrow \mathbb{R}_n[X]$ are defined by the following prescriptions:

$$\begin{aligned} T_w 1 &= 1 \\ T_w(X^k) &= \sum_{h=0}^{k-1} \frac{k!}{(k+1-h)!h!} w^{k-h} X^h + X^k; \quad \forall k \in \{1, \dots, n\} \\ (S_u P)(X) &:= P(X + u) \quad \text{translation operator on } \mathbb{R}_n[X] \end{aligned}$$

Denote

$\mathcal{W}_{F,1,n} :=$ norm closure in $\mathcal{B}(\Gamma(\mathbb{C}))$ of the linear span of the operators (7).

Identity (8) implies that $\mathcal{W}_{F,1,n}$ is a C^* -algebra.

In [10] it is proved that the composition law (9) is a Lie group law on $\mathbb{R} \times \mathbb{R}_n[X]$ whose Lie algebra is $heis_{\mathbb{R}}(1, n)$. Since the elements of $heis_{\mathbb{R}}(1, n)$ are parametrized by the pairs $(u, P) \in \mathbb{R} \times \mathbb{R}_n[X]$ it is natural to introduce the following notation.

Definition 2 (see [10]) The 1-mode n -th degree Heisenberg group is the set

$$\text{Heis}(1, n) := \{e^{\ell_0(u, P)} : (u, P) \in \mathbb{R} \times \mathbb{R}_n[X]\} \quad (10)$$

with composition law

$$e^{\ell_0(u, P)} \circ e^{\ell_0(v, Q)} := e^{\ell_0((u+v, T_{u+v}^{-1}(T_u P + T_v S_u Q)))}$$

The name $\text{Heis}(1, n)$ is motivated by the fact that, for $n = 1$, $\text{Heis}(1, n)$ reduces to the usual the 1-mode Heisenberg group.

4 The free group- C^* -algebra of $\text{Heis}(1, n)$

Definition 3 Let G be a group. The free complex vector space generated by the set

$$\{W_g : g \in G\}$$

has a unique structure of unital $*$ -algebra defined by the prescription that the map $g \mapsto W_g$ defines a unitary representation of G , equivalently:

$$\begin{aligned} W_g W_h &:= W_{gh} & ; & \quad g, h \in G \\ (W_g)^* &:= W_{g^{-1}} & ; & \quad g \in G \\ 1 &:= W_e \end{aligned} \tag{11}$$

The completion of $\mathcal{W}^0(G)$ under the (minimal) C^* -norm

$$\|x\| := \sup\{\|\pi(x)\| : \pi \in \{\text{*}-\text{representations of } G\}\} \quad ; \quad x \in \mathcal{W}^0(G)$$

will be called the *free group- C^* -algebra of G* and denoted $\mathcal{W}(G)$.

Remark 2 Because of (11) a $*$ -representation of $\mathcal{W}(G)$ maps the generators W_g ($g \in G$), into unitary operators.

Remark 3 If G, G' are groups, then any group homomorphism (resp. isomorphism) $\alpha : G \rightarrow G'$, extends uniquely to a C^* -algebra homomorphism (resp. isomorphism) $\tilde{\alpha} : \mathcal{W}(G) \rightarrow \mathcal{W}(G')$ characterized by the condition

$$\tilde{\alpha}(W_g) := W_{\alpha g} \quad ; \quad g \in G$$

Definition 4 If $G = \text{Heis}(1, n)$, its free group- C^* -algebra is called the *1-mode n -th degree Weyl algebra* and denoted

$$\mathcal{W}_{1,n}^0 := \mathcal{W}^0(\text{Heis}(1, n)) \tag{12}$$

For its generators, called the *1-mode n -th degree Weyl operators*, we will use the notation

$$W^0(u, P) := W_{e^{\ell_0(u, P)}} \quad ; \quad (u, P) \in \mathbb{R} \times \mathbb{R}_n[X] \tag{13}$$

By construction the map

$$u_F : W^0(u, P) \in \mathcal{W}_{1,n}^0 \mapsto W(u, P) \in \mathcal{W}_{F,1,n} \tag{14}$$

where the operators $W(u, P)$ are those defined in (7), is a group isomorphism. Hence the definition of free group- C^* -algebra implies that it can be extended to a surjective $*$ -representation called *the Fock representation of $\mathcal{W}_{1,n}^0$* . We will use the same symbol u_F for this extension.

We conjecture that, in analogy with the case $n = 1$, the $*$ -homomorphism of $\mathcal{W}_{1,n}^0$ onto $\mathcal{W}_{F,1,n}$ is in fact an isomorphism and that there is a unique C^* -norm on $\mathcal{W}_{1,n}^0$.

5 The current algebra of $heis_{\mathbb{R}}(1, n)$ over \mathbb{R}

Denote

$$\mathcal{H}_0(\mathbb{R}) := L_{\mathbb{R}}^1(\mathbb{R}) \cap L_{\mathbb{R}}^\infty(\mathbb{R}) = \bigcap_{1 \leq p \leq \infty} L_{\mathbb{R}}^p(\mathbb{R})$$

$\mathcal{H}_0(\mathbb{R})$ has a natural structure of real pre-Hilbert algebra with the pointwise operations and the L^2 -scalar product.

Lemma 2 For any $*$ -sub-algebra \mathcal{T} of $\mathcal{H}_0(\mathbb{R})$ and $n \in \mathbb{N}$, there exists a unique real $*$ -Lie algebra with skew-adjoint generators

$$\{L_0, L_k(f) : k \in \{1, \dots, n+1\} ; f \in \mathcal{T}\}$$

where, with the notation

$$L_0(f) := L_0 \int_{\mathbb{R}} f(t) dt \quad ; \quad L_{-1}(f) = 0 \quad ; \quad \forall f \in \mathcal{T} \quad (15)$$

the maps $f \mapsto L_k(f)$ ($k \in \{0, 1, \dots, n\}$) are real linear on \mathcal{T} and the Lie brackets are given, for all $f, g \in \mathcal{T}$, by

$$[L_i(f), L_j(g)] = 0 \quad ; \quad i, j \in \{0, 1, \dots, n\} \quad (16)$$

$$[L_{n+1}(f), L_k(g)] = kL_{k-1}(fg) ; k \in \{0, 1, 2, \dots, n\} , L_{-1}(f) = 0 \quad (17)$$

Proof. By definition the Lie brackets of two generators defined by (16), (17) is a multiple of the generators. In order to verify that the Jacobi identity is satisfied notice that, for any $i, j, k \in \{0, 1, \dots, n\}$

$$[L_i(f_1), [L_j(f_2), L_k(f_3)]] = 0$$

unless exactly 2 among the indices i, j, k are equal to $n+1$. Moreover, up to change of sign one can assume that $i = j = n+1$. In this case one verifies that

$$\begin{aligned} [L_{n+1}(f_1), [L_{n+1}(f_2), L_k(f_3)]] &= k(k-1)L_{k-2}(f_1f_2f_3) \\ [L_{n+1}(f_2), [L_k(f_3), L_{n+1}(f_1)]] &= -k(k-1)L_{k-2}(f_1f_2f_3) \\ [L_k(f_3), [L_{n+1}(f_1), L_{n+1}(f_2)]] &= 0 \end{aligned}$$

and adding these identities side by side the Jacobi identity follows. \square

Definition 5 The real \ast -Lie algebra defined in Lemma 2 will be denoted $heis_{\mathbb{R}}(1, n, \mathcal{T})$. If $I \subset \mathbb{R}$ is a bounded Borel subsets we denote

$$\mathcal{T}_I := \text{the sub-algebra of } \mathcal{T} \text{ of functions with support in } I \quad (18)$$

In analogy with the notation (3) we write the generic element of $heis_{\mathbb{R}}(1, n, \mathcal{T})$ in the form

$$\ell(\tilde{f}) := L_{n+1}(f_{n+1}) + \sum_{k=0}^n L_k(f_k) \quad ; \quad f_0, \dots, f_{n+1} \in \mathcal{T} \quad (19)$$

where, here and in the following, if (f_0, \dots, f_{n+1}) is an ordered $(n+2)$ -uple of elements of \mathcal{T} , we will use the notation

$$\tilde{f} := (f_0, \dots, f_{n+1}) \quad (20)$$

6 Isomorphisms between the current algebra $heis_{\mathbb{R}}(1, n, \mathbb{R}\chi_I)$ and $heis_{\mathbb{R}}(1, n)$

In the notations of the previous section and of Definition 5, for a bounded Borel subset I of \mathbb{R} , we denote

$$\chi_J(x) := \begin{cases} 1 & \text{if } x \in J \\ 0 & \text{if } x \notin J \end{cases}$$

$$\mathbb{R}\chi_I := \{\text{the real algebra of multiples of } \chi_I\}$$

Thus

$$heis_{\mathbb{R}}(1, n, \mathbb{R}\chi_I) \subset heis_{\mathbb{R}}(1, n, \mathcal{H}_0(\mathbb{R}))$$

is the \ast -Lie sub-algebra of $heis_{\mathbb{R}}(1, n, \mathcal{H}_0(\mathbb{R}))$ with linear skew-adjoint generators

$$\{L_k(\chi_I) \quad : \quad k \in \{0, 1, \dots, n\}\}$$

and brackets

$$[L_{n+1}(\chi_I), L_k(\chi_I)] = kL_{k-1}(\chi_I) \quad ; \quad k \in \{0\} \cup \{2, \dots, n\} \quad (21)$$

for $k \in \{2, \dots, n\}$ and the other commutators vanish. Recalling the notation (15) one must have

$$L_0(\chi_I) = |I|L_0$$

Lemma 3 In the notations of section 3 a real linear map $\hat{s}_I : \text{heis}_{\mathbb{R}}(1, n, \mathbb{R}\chi_I) \rightarrow \text{heis}_{\mathbb{R}}(F, 1, n)$ satisfying for some constants $a_I, b_I, c_{k,I} \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and for each $k \in \{1, \dots, n\}$

$$\hat{s}_I(L_0) = a_I i 1 \quad (22)$$

$$\hat{s}_I(L_{n+1}(\chi_I)) = b_I i p \quad (23)$$

$$\hat{s}_I(L_k(\chi_I)) = c_{k,I} i q^k \quad ; \quad \forall k \in \{1, \dots, n\} \quad (24)$$

is a real $*$ -Lie algebra isomorphism if and only if

$$c_{k,I} = b_I^{-k} |I| a_I \quad ; \quad \forall k \in \{1, \dots, n\} \quad (25)$$

The additional condition

$$c_{1,I} = b_I \quad (26)$$

implies that a_I must be > 0 and:

$$c_{k,I} = |I|^{1-\frac{k}{2}} a_I^{1-\frac{k}{2}} \quad ; \quad \forall k \in \{1, \dots, n\} \quad (27)$$

Remark 4 In the above statement $\text{heis}_{\mathbb{R}}(F, 1, n)$ can be replaced by $\text{heis}_{\mathbb{R}}(1, n)$ because of the real $*$ -Lie algebra isomorphism between the two.

Proof. By definition \hat{s}_I maps a basis of $\text{heis}_{\mathbb{R}}(1, n, \mathbb{R}\chi_I)$ into a basis of $\text{heis}_{\mathbb{R}}(F, 1, n)$ because the constants $b_I, c_{k,I}$ are non zero hence it defines a unique vector space isomorphism which is a $*$ -map because the constants are real. Moreover (22), (23), and (24) imply that

$$[\hat{s}_I(L_{n+1}(\chi_I), \hat{s}_I(L_1(\chi_I)))] = [b_I i p, c_{1,I} i q] = b_I c_{1,I} [i p, i q] = b_I c_{1,I} i 1$$

while (21) and (24) imply that

$$\hat{s}_I([L_{n+1}(\chi_I), L_1(\chi_I)]) = \hat{s}_I(|I| L_0) = |I| \hat{s}_I(L_0) = |I| a_I i 1$$

The isomorphism condition then implies that

$$b_I c_{1,I} = |I| a_I \quad (28)$$

The same argument, using (21), shows that for all $k \in \{2, \dots, n\}$

$$[\hat{s}_I(L_{n+1}(\chi_I)), \hat{s}_I(L_k(\chi_I))] = [b_I i p, c_{k,I} i q^k] = b_I c_{k,I} [i p, i q^k] = b_I c_{k,I} k i q^{k-1}$$

$$\hat{s}_I([L_{n+1}(\chi_I), L_k(\chi_I)]) = \hat{s}_I(kL_{k-1}(\chi_I)) = k\hat{s}_I(L_{k-1}(\chi_I)) = kc_{k-1,I}iq^{k-1}$$

and the isomorphism condition implies that

$$b_I c_{k,I} = c_{k-1,I} \Leftrightarrow c_{k,I} = b_I^{-1} c_{k-1,I} = b_I^{-2} c_{k-2,I} = \dots = b_I^{-(k-1)} c_{1,I} = b_I^{-k} |I| a_I$$

which is (25). Finally, if (26) holds, then (28) becomes

$$b_I^2 = |I| a_I$$

Thus a_I must be > 0 and $b_I = |I|^{1/2} a_I^{1/2}$ which implies (27). \square

Remark 5 In the following we fix condition (26) and put

$$a_I = 1 \tag{29}$$

for all I so that the real \ast -Lie algebras isomorphism \hat{s}_I is given by (22) and (27). Therefore its inverse \hat{s}_I^{-1} is given, on the generators, by:

$$\begin{aligned} \hat{s}_I^{-1}(i1) &= L_0 \\ \hat{s}_I^{-1}(ip) &= |I|^{-\frac{1}{2}} L_{n+1}(\chi_I) \\ \hat{s}_I^{-1}(iq^k) &= |I|^{k/2-1} L_k(\chi_I) \quad ; \quad \forall k \in \{1, \dots, n\} \end{aligned}$$

The reason why the additional conditions (25) and (29) are necessary will be explained in Remark 7 at the end of section 9.

Remark 6 *Lemma 3 and condition (29) mean that, for any bounded Borel set $I \subset \mathbb{R}$, $heis_{\mathbb{R}}(1, n, \mathbb{R}\chi_I)$ can be identified to a copy of $heis_{\mathbb{R}}(1, n)$ with the rescaled basis*

$$\{i|I|L_0 \quad , \quad i|I|^{\frac{1}{2}}L_{n+1} \quad , \quad i|I|^{1-\frac{k}{2}}L_k \quad , \quad k = 1, \dots, n\} \tag{30}$$

In analogy with (3), we parametrize the elements of $heis_{\mathbb{R}}(1, n, \mathbb{R}\chi_I)$, with elements of $\mathbb{R} \times \mathbb{R}_n[X]$, and we write

$$\ell_I(u, P) := uL_{n+1}(\chi_I) + P(L(\chi_I)) \quad ; \quad u \in \mathbb{R} \tag{31}$$

where $P := \sum_{j=0}^n a_j X^j$ is a polynomial in one indeterminate and we use the convention

$$P(L(\chi_I)) := \sum_{j=0}^n a_j L_j(\chi_I) := a_0 |I| L_0 + \sum_{j=1}^n a_j L_j(\chi_I) \tag{32}$$

The image of such an element under the isomorphism \hat{s}_I is

$$\hat{s}_I(\ell_I(u, P)) = i(u|I|^{\frac{1}{2}}p + P_I(q)) \quad (33)$$

where by definition:

$$P_I(X) := \sum_{j=0}^n a_j |I|^{1-\frac{j}{2}} X^j = a_0 |I| + \sum_{j=1}^n a_j |I|^{1-\frac{j}{2}} X^j \quad (34)$$

Introducing the linear change of coordinates in $\mathbb{R} \times \mathbb{R}_n[X]$ defined by

$$\hat{k}_I(u, P) := (u|I|^{\frac{1}{2}}, P_I) \equiv \left(u|I|^{\frac{1}{2}}, (a_j |I|^{1-\frac{j}{2}}) \right) \quad (35)$$

where P_I is defined by (34) we see that, in the notations (3) and (31) one has

$$\hat{s}_I \circ \ell_I = \ell_0 \circ \hat{k}_I \quad (36)$$

7 The group $\text{Heis}(1, n, \mathbb{R}_{\chi_I})$ and its C^* -algebra

In the notations and assumptions of section 6 we have seen that $\text{heis}_{\mathbb{R}}(1, n, \mathbb{R}_{\chi_I})$ is isomorphic to $\text{heis}_{\mathbb{R}}(1, n)$. Since \mathbb{R}^{n+2} is connected and simply connected, the Lie group of $\text{heis}_{\mathbb{R}}(1, n, \mathbb{R}_{\chi_I})$, denoted $\text{Heis}(1, n, \mathbb{R}_{\chi_I})$ is isomorphic to $\text{Heis}(1, n)$. In analogy with the notation (10), the generic element of $\text{Heis}(1, n, \mathbb{R}_{\chi_I})$ will be denoted

$$e^{\ell_I(u, P)} \quad ; \quad (u, P) \in \mathbb{R} \times \mathbb{R}_n[X] \quad (37)$$

Definition 6 For any bounded Borel set $I \subset \mathbb{R}$ we denote

$$\mathcal{W}_{1,n;I}^0 := \mathcal{W}(\text{Heis}(1, n, \mathbb{R}_{\chi_I}))$$

the free group- C^* -algebra of the group $\text{Heis}(1, n, \mathbb{R}_{\chi_I})$. In analogy with (13), its generators will be called the *one mode n -th degree Weyl operators localized on I* and denoted

$$W_I^0(u, P) := W_{e^{\ell_I(u, P)}} \in \mathcal{W}_{1,n;I}^0 \quad (38)$$

Remark 7 Since the groups $\text{Heis}(1, n, \mathbb{R}\chi_I)$ and $\text{Heis}(1, n)$ are isomorphic, the same is true for the corresponding free group- C^* -algebras.

In the following section we show that, in these C^* -algebra isomorphisms, the group generators of $\mathcal{W}_{1,n;I}^0$ are mapped into a set of group generators of $\mathcal{W}_{1,n}^0$ which depends on I and we introduce a construction that allows to get rid of this dependence.

7.1 C^* -algebras isomorphism

In the notations (3) and (37) the map

$$e^{\ell_I(u,P)} \in \text{Heis}(1, n, \mathbb{R}\chi_I) \mapsto e^{\hat{s}_I(\ell_I(u,P))} \in \text{Heis}(1, n)$$

where \hat{s}_I the isomorphism defined in Lemma 3, is a Lie group isomorphism, hence it can be extended to a C^* -isomorphism of the corresponding free group- C^* -algebras.

This extension will be denoted with the symbol:

$$s_I^0 : \mathcal{W}_{1,n;I}^0 \rightarrow \mathcal{W}_{1,n}^0$$

In view of the identity (36), and in the notations (13) and (38), the explicit form of s_I^0 on the generators is given by

$$s_I^0(W_I^0(u, P)) = W^0(\hat{k}_I(u, P)) \quad (39)$$

where \hat{k}_I is the linear map defined by (35) and $(u, P) \in \mathbb{R} \times \mathbb{R}_n[X]$.

It is clear from (35) and (39) that, as a vector space, $s_I^0(\mathcal{W}_{1,n;I}^0)$ coincides with $\mathcal{W}_{1,n}^0$. In this section we will prove that the map

$$W^0(u, P) \in \mathcal{W}_{1,n}^0 \mapsto W^0(\hat{k}_I(u, P)) \in \mathcal{W}_{1,n}^0 \quad (40)$$

induces a C^* -algebra automorphism denoted k_I . To this goal we use

$$W^0(\hat{k}_I(u, P))W^0(\hat{k}_I(v, Q)) = W^0(\hat{k}_I(u, P) \circ \hat{k}_I(v, Q))$$

and the following result.

Lemma 4 For all $u \in \mathbb{R}$ and $P \in \mathbb{R}_n[X]$, let \hat{k}_I be the linear map defined by (35). Then, denoting with the same symbol \hat{k}_I its restriction on $\mathbb{R}_n[X]$, one has:

$$\begin{aligned}\hat{k}_I \circ T_u(P) &= T_{u|I|^{\frac{1}{2}}} \circ \hat{k}_I(P) \\ \hat{k}_I^{-1} \circ T_u^{-1}(P) &= T_{u|I|^{-\frac{1}{2}}}^{-1} \circ \hat{k}_I^{-1}(P) \\ \hat{k}_I \circ T_u^{-1}(P) &= T_{u|I|^{\frac{1}{2}}}^{-1} \circ \hat{k}_I(P) \\ \hat{k}_I^{-1} \circ T_u(P) &= T_{u|I|^{-\frac{1}{2}}} \circ \hat{k}_I^{-1}(P)\end{aligned}$$

Proof. Since both T_u and \hat{k}_I are linear maps, it is sufficient to prove the lemma for $P(X) = X^k$ ($k \in \{0, \dots, n\}$). For $k = 0$ all the identities in the lemma are obviously true. Let $k \in \{1, \dots, n\}$. Then from the identity (10) one has

$$\begin{aligned}T_{u|I|^{\frac{1}{2}}} \circ \hat{k}_I(X^k) &= T_{u|I|^{\frac{1}{2}}}(|I|^{1-\frac{k}{2}}X^k) \\ &= |I|^{1-\frac{k}{2}} T_{u|I|^{\frac{1}{2}}}(X^k) \\ &= |I|^{1-\frac{k}{2}} \left[\sum_{h=0}^{k-1} \frac{k!}{(k+1-h)!h!} u^{k-h} |I|^{\frac{k-h}{2}} X^h + X^k \right] \\ &= \sum_{h=0}^{k-1} \frac{k!}{(k+1-h)!h!} u^{k-h} |I|^{1-\frac{h}{2}} X^h + |I|^{1-\frac{k}{2}} X^k \\ &= \hat{k}_I \circ T_u(X^k)\end{aligned} \tag{41}$$

(41) is equivalent to

$$T_{u|I|^{\frac{1}{2}}} \circ \hat{k}_I = \hat{k}_I \circ T_u \Leftrightarrow \hat{k}_I^{-1} \circ T_{u|I|^{\frac{1}{2}}}^{-1} = T_u^{-1} \circ \hat{k}_I^{-1}$$

Replacing u by $u|I|^{-\frac{1}{2}}$, this yields

$$\hat{k}_I^{-1} \circ T_u^{-1} = T_{u|I|^{-\frac{1}{2}}}^{-1} \circ \hat{k}_I^{-1} \tag{42}$$

From identities (41) and (42), one gets

$$\begin{aligned}T_{u|I|^{\frac{1}{2}}} \circ \hat{k}_I \circ T_u^{-1} &= \hat{k}_I \\ T_{u|I|^{-\frac{1}{2}}} \circ \hat{k}_I^{-1} \circ T_u^{-1} &= \hat{k}_I^{-1}\end{aligned}$$

or equivalently

$$\begin{aligned}\hat{k}_I \circ T_u^{-1} &= T_{u|I|^{\frac{1}{2}}}^{-1} \circ \hat{k}_I \\ \hat{k}_I^{-1} \circ T_u &= T_{u|I|^{-\frac{1}{2}}} \circ \hat{k}_I^{-1}\end{aligned}$$

□

Proposition 1 \hat{k}_I is a group automorphism for the composition law (9).

Proof. We have to prove that for all $(u, P), (v, Q) \in \mathbb{R} \times \mathbb{R}_n[X]$, one has

$$(\hat{k}_I(u, P) \circ \hat{k}_I(v, Q)) = \hat{k}_I\left((u+v); T_{(u+v)}^{-1}(T_u P + T_v S_u Q)\right)$$

We know that

$$\hat{k}_I(u, P) \circ \hat{k}_I(v, Q) = (u|I|^{\frac{1}{2}}, P_I) \circ (v|I|^{\frac{1}{2}}, Q_I)$$

where $P_I(X) = P(|I|^{-\frac{1}{2}}X)$ and $Q_I(X) = Q(|I|^{-\frac{1}{2}}X)$. But from (11) we know that

$$\begin{aligned}(u|I|^{\frac{1}{2}}, P_I) \circ (v|I|^{\frac{1}{2}}, Q_I) &= \left((u+v)|I|^{\frac{1}{2}}, T_{(u+v)|I|^{\frac{1}{2}}}^{-1}(T_{u|I|^{\frac{1}{2}}} P_I + T_{v|I|^{\frac{1}{2}}} S_u Q_I)\right) \\ &= \left((u+v)|I|^{\frac{1}{2}}, T_{(u+v)|I|^{\frac{1}{2}}}^{-1}(T_{u|I|^{\frac{1}{2}}} \hat{k}_I(P) + T_{v|I|^{\frac{1}{2}}} S_u \hat{k}_I(Q))\right)\end{aligned}$$

Furthermore, from Lemma 4, we know that

$$T_{(u+v)|I|^{\frac{1}{2}}}^{-1} T_{u|I|^{\frac{1}{2}}} \hat{k}_I(P) = \hat{k}_I T_{(u+v)}^{-1} T_u(P)$$

Moreover, using

$$S_u \hat{k}_I(Q) = \hat{k}_I S_u(Q)$$

We also have

$$\begin{aligned}T_{(u+v)|I|^{\frac{1}{2}}}^{-1} T_{v|I|^{\frac{1}{2}}} S_u \hat{k}_I(Q) &= T_{(u+v)|I|^{\frac{1}{2}}}^{-1} T_{v|I|^{\frac{1}{2}}} \hat{k}_I S_u(Q) \\ &= T_{(u+v)|I|^{\frac{1}{2}}}^{-1} \hat{k}_I T_v S_u(Q) \\ &= \hat{k}_I T_{(u+v)}^{-1} T_v S_u(Q)\end{aligned}$$

Hence, one gets

$$\begin{aligned}
(u|I|^{\frac{1}{2}}, P_I) \circ (v|I|^{\frac{1}{2}}, Q_I) &= ((u+v)|I|^{\frac{1}{2}}, \hat{k}_I T_{(u+v)}^{-1}(T_u(P) + T_v S_u(Q))) \\
&= \hat{k}_I \left((u+v), T_{(u+v)}^{-1}(T_u(P) + T_v S_u(Q)) \right) \\
&= \hat{k}_I((u, P) \circ (v, Q))
\end{aligned}$$

and this proves the statement. \square

Corollary 1 The map:

$$s_I := k_I^{-1} \circ s_I^0 : \mathcal{W}_{1,n;I}^0 \rightarrow \mathcal{W}_{1,n}^0$$

is a C^* -algebra isomorphism characterized by the condition

$$s_I(W_I^0(u, P)) = W^0(u, P) \quad ; \quad \forall (u, P) \in \mathbb{R}^{n+2} \quad (43)$$

Proof. (43) is clear from (39) and the definition (40) of k_I .

We know that s_I^0 is a C^* -algebra isomorphism. From Proposition 1 we know that \hat{k}_I is a group automorphism for the composition law defined by (9). Because of the linear independence of the free group algebra generators k_I extends to a C^* -algebra automorphism. Thus s_I is composed of an isomorphism with an automorphism and the thesis follows. \square

8 The inductive limit

In the following, when speaking of tensor products of C^* -algebras, it will be understood that a choice of a cross norm has been fixed and that all tensor products are referred to the same choice.

For a bounded Borel subset I of \mathbb{R} , let $\mathcal{W}_{1,n;I}^0$ be the C^* -algebra in Definition 6 and let the isomorphisms $s_I : \mathcal{W}_{1,n;I}^0 \rightarrow \mathcal{W}_{1,n}^0$ defined by (43). For $\pi = (I_j)_{j \in F} \in Part_{fin}(I)$ define the C^* -algebra

$$\mathcal{W}_{1,n;I;\pi}^0 := \bigotimes_{j \in F} \mathcal{W}_{1,n;I_j}^0 \quad (44)$$

the injective C^* -homomorphism (C^* -embedding)

$$z_{I,\pi} := \left(\begin{smallmatrix} (diag) \\ \bigotimes_{j \in F} s_{I_j}^{-1} \end{smallmatrix} \right) \circ s_I : \mathcal{W}_{1,n;I}^0 \rightarrow \bigotimes_{j \in F} \mathcal{W}_{1,n;I_j}^0 = \mathcal{W}_{1,n;I;\pi}^0$$

Then, for any $\pi \prec \pi' \in Part_{fin}(I)$, the map

$$z_{I;\pi,\pi'} := \bigotimes_{j \in F} \left(\bigotimes_{I_j \supseteq I' \in \pi'}^{(diag)} s_{I'}^{-1} \right) \circ s_{I_j} : \mathcal{W}_{1,n;I;\pi}^0 \rightarrow \bigotimes_{j \in F} \bigotimes_{I_j \supseteq I' \in \pi'} \mathcal{W}_{1,n;I'}^0 = \mathcal{W}_{1,n;I;\pi'}^0 \quad (45)$$

is a C^* -embedding. Moreover, by construction and in the notations of Definition 6, for all $u \in \mathbb{R}$ and $P = \sum_{j=0}^n a_j X^j \in \mathbb{R}_n[X]$, one has

$$z_{I;\pi,\pi'} z_{I,\pi}(W_I^0(u, P)) := \bigotimes_{I' \in \pi'} W_{I'}^0(u, P) = z_{I,\pi'}(W_I^0(u, P)) \in \mathcal{W}_{1,n;I;\pi'}^0$$

Lemma 5 The family

$$\left\{ (\mathcal{W}_{1,n;I;\pi}^0)_{\pi \in Part_{fin}(I)} , (z_{I;\pi,\pi'})_{\pi \prec \pi' \in Part_{fin}(I)} \right\} \quad (46)$$

is an inductive system of C^* -algebras, i.e. for all $\pi \prec \pi'$, $z_{I;\pi,\pi'}$ is a morphism and if $\pi \prec \pi' \prec \pi'' \in Part_{fin}(I)$ one has

$$z_{I;\pi',\pi''} z_{I;\pi,\pi'} = z_{I;\pi,\pi''} \quad (47)$$

Proof. We have already proved that the $z_{I;\pi,\pi'}$ are C^* -embeddings. Therefore it remains to prove (47). To this goal, for π, π', π'' as in the statement, using the identity

$$\bigotimes_{I' \in \pi'} = \bigotimes_{I \in \pi} \bigotimes_{\pi' \ni I' \subseteq I}$$

one finds

$$\begin{aligned} z_{I;\pi'',\pi'} z_{I;\pi,\pi'} &= \left(\bigotimes_{I' \in \pi'} \left(\bigotimes_{\pi'' \ni I'' \subseteq I'}^{(diag)} s_{I''}^{-1} \right) \circ s_{I'} \right) \left(\bigotimes_{I \in \pi} \left(\bigotimes_{\pi' \ni I' \subseteq I}^{(diag)} s_{I'}^{-1} \right) \circ s_I \right) \\ &= \left(\bigotimes_{I \in \pi} \bigotimes_{\pi' \ni I' \subseteq I} \left(\bigotimes_{\pi'' \ni I'' \subseteq I'}^{(diag)} s_{I''}^{-1} \right) \circ s_{I'} \right) \left(\bigotimes_{I \in \pi} \left(\bigotimes_{\pi' \ni I' \subseteq I}^{(diag)} s_{I'}^{-1} \right) \circ s_I \right) \\ &= \bigotimes_{I \in \pi} \left(\bigotimes_{\pi' \ni I' \subseteq I} \left(\bigotimes_{\pi'' \ni I'' \subseteq I'}^{(diag)} s_{I''}^{-1} \right) \circ s_{I'} \right) \left(\left(\bigotimes_{\pi' \ni I' \subseteq I}^{(diag)} s_{I'}^{-1} \right) \circ s_I \right) \end{aligned}$$

$$\begin{aligned}
&= \bigotimes_{I \in \pi} \bigotimes_{\pi' \ni I' \subseteq I} \left(\left(\bigotimes_{\pi'' \ni I'' \subseteq I'} \overset{(diag)}{\bigotimes} s_{I''}^{-1} \right) \circ s_{I'} (s_{I'}^{-1}) \circ s_I \right) \\
&= \bigotimes_{I \in \pi} \left(\left(\bigotimes_{\pi' \ni I' \subseteq I} \bigotimes_{\pi'' \ni I'' \subseteq I'} \overset{(diag)}{\bigotimes} s_{I''}^{-1} \right) \circ s_I \right) \\
&= \bigotimes_{I \in \pi} \left(\left(\bigotimes_{\pi'' \ni I'' \subseteq I} \overset{(diag)}{\bigotimes} s_{I''}^{-1} \right) \circ s_I \right) = z_{I; \pi'', \pi}
\end{aligned}$$

□

Definition 7 For any bounded Borel subset I of \mathbb{R} , we denote

$$\{\mathcal{W}_{1,n;I}, (\tilde{z}_{I;\pi})_{\pi \in Part_{fin}(I)}\}$$

the inductive limit of the family (46) i.e., $\mathcal{W}_{1,n;I}$ is a C^* -algebra and for any $\pi \in Part_{fin}(I)$ and in the notation (44),

$$\tilde{z}_{I;\pi} : \mathcal{W}_{1,n;I;\pi}^0 \rightarrow \mathcal{W}_{1,n;I}$$

is an embedding satisfying

$$\tilde{z}_{I;\pi'} z_{I;\pi, \pi'} = \tilde{z}_{I;\pi} \quad ; \quad \forall \pi \prec \pi' \in Part_{fin}(I)$$

Remark Intuitively one can think of the elements of $\mathcal{W}_{1,n;I}$ as a realization of the *non-linear Weyl operators*: (1) with finitely valued, compact support, test functions.

8.1 Factorizable families of C^* -algebras

Definition 8 A family of C^* -algebras $\{\mathcal{W}_I\}$, indexed by the bounded Borel subsets of \mathbb{R} , is called *factorizable* if, for every bounded Borel $I \subset \mathbb{R}$ and every Borel partition π of I , there is an isomorphism

$$u_{I,\pi} : \bigotimes_{I_j \in \pi} \mathcal{W}_{I_j} \rightarrow \mathcal{W}_I$$

If this is the case, an operator $w_I \in \mathcal{W}_I$ is called *factorizable* if there exist operators $w_{I_j} \in \mathcal{W}_{I_j}$ ($I_j \in \pi$) such that

$$u_{I,\pi}^{-1}(w_I) = \bigotimes_{I_j \in \pi} w_{I_j} \tag{48}$$

Remark 8 *In the following, for a given bounded Borel set I , when $\pi \equiv \{I\}$ is the partition of I , consisting of the only set I , we will use the notation*

$$\tilde{z}_I := \tilde{z}_{I;\{I\}} : \mathcal{W}_{1,n;I}^0 \rightarrow \mathcal{W}_{1,n;I}$$

We want to prove that:

(i) the family of C^* -algebras

$$\{\mathcal{W}_{1,n;I} : I\text{-bounded Borel subset of } \mathbb{R}\} \quad (49)$$

where the algebras $\mathcal{W}_{1,n;I}$ are those introduced in Definition 7, is factorizable in the sense of Definition 8;

(ii) for any bounded Borel set I , the operators

$$W_I(u, P) := \tilde{z}_I(W_I^0(u, P)) \in \mathcal{W}_{1,n;I} \quad ; \quad W_I^0(u, P) \in \mathcal{W}_{1,n;I}^0 \quad (50)$$

are factorizable in the sense of (48).

To this goal let us remark that, if I, J are disjoint bounded Borel sets in \mathbb{R} , then the map

$$(\pi_I, \pi_J) \in Part_{fin}(I) \times Part_{fin}(J) \mapsto \pi_{I \cup J} := \{\pi_I \cup \pi_J\} \in Part_{fin}(I \cup J) \quad (51)$$

defines a canonical bijection between $Part_{fin}(I) \times Part_{fin}(J)$ and $Part_{fin}(I \cup J)$ such that, if $\pi_I \prec \pi'_I \in Part_{fin}(I)$ and $\pi_J \prec \pi'_J \in Part_{fin}(J)$, then $\pi_{I \cup J} \prec \pi'_{I \cup J} \in Part_{fin}(I \cup J)$.

Lemma 6 *Let I, J be disjoint bounded Borel sets in \mathbb{R} . Then the inductive system of C^* -algebras*

$$\{(\mathcal{W}_{1,n;I \cup J; \pi_{I \cup J}}^0)_{\pi_{I \cup J} \in Part_{fin}(I \cup J)}, (z_{I \cup J; \pi_{I \cup J}, \pi'_{I \cup J}})_{\pi_{I \cup J} \prec \pi'_{I \cup J} \in Part_{fin}(I \cup J)}\} \quad (52)$$

is isomorphic to the inductive system of C^ -algebras*

$$\begin{aligned} & \{(\mathcal{W}_{1,n;I; \pi_I}^0 \otimes \mathcal{W}_{1,n;J; \pi_J}^0)_{(\pi_I, \pi_J) \in Part_{fin}(I) \times Part_{fin}(J)}, \\ & (z_{I; \pi_I, \pi'_I} \otimes z_{J; \pi_J, \pi'_J})_{\pi_I \prec \pi'_I \in Part_{fin}(I), \pi_J \prec \pi'_J \in Part_{fin}(J)}\} \end{aligned} \quad (53)$$

in the sense that, for each $\pi_I \in \text{Part}_{fin}(I)$ and $\pi_J \in \text{Part}_{fin}(J)$, then there exists a C^* -algebra isomorphism

$$u_{I,J,\pi_I,\pi_J} : \mathcal{W}_{1,n;I;\pi_I}^0 \otimes \mathcal{W}_{1,n;J;\pi_J}^0 \rightarrow \mathcal{W}_{1,n;I \cup J;\pi_{I \cup J}}^0$$

such that, for each $\pi_I \prec \pi'_I \in \text{Part}_{fin}(I)$ and $\pi_J \prec \pi'_J \in \text{Part}_{fin}(J)$, one has in the notation (51)

$$u_{I,J,\pi_I,\pi_J} \circ (z_{I;\pi_I,\pi'_I} \otimes z_{J;\pi_J,\pi'_J}) = z_{I \cup J;\pi_{I \cup J},\pi'_{I \cup J}} \quad (54)$$

Proof. With the notations above, from (44) one deduces that

$$\begin{aligned} \mathcal{W}_{1,n;I;\pi_I}^0 \otimes \mathcal{W}_{1,n;J;\pi_J}^0 &:= \left(\bigotimes_{I' \in \pi_I} \mathcal{W}_{1,n;I'}^0 \right) \otimes \left(\bigotimes_{J' \in \pi_J} \mathcal{W}_{1,n;J'}^0 \right) \\ &\equiv \bigotimes_{K \in \pi_{I \cup J}} \mathcal{W}_{1,n;K}^0 = \mathcal{W}_{1,n;I \cup J;\pi_{I \cup J}}^0 \end{aligned} \quad (55)$$

Denote

$$u_{I \otimes J, I \cup J} : \mathcal{W}_{1,n;I;\pi_I}^0 \otimes \mathcal{W}_{1,n;J;\pi_J}^0 \rightarrow \mathcal{W}_{1,n;I \cup J;\pi_{I \cup J}}^0$$

the isomorphism defined by (55). If $\pi_I \prec \pi'_I \in \text{Part}_{fin}(I)$ and $\pi_J \prec \pi'_J \in \text{Part}_{fin}(J)$, then clearly $\pi_{I \cup J} \prec \pi'_{I \cup J} \in \text{Part}_{fin}(I \cup J)$ and from (45) we see that

$$\begin{aligned} &u_{I,J,\pi_I,\pi_J} \circ (z_{I;\pi_I,\pi'_I} \otimes z_{J;\pi_J,\pi'_J}) \\ &= u_{I,J,\pi_I,\pi_J} \circ \left(\left(\bigotimes_{I_j \in \pi_I} \left(\bigotimes_{I_j \supseteq I' \in \pi'_I} \begin{pmatrix} (diag) \\ s_{I'}^{-1} \end{pmatrix} \right) \circ s_{I_j} \right) \otimes \left(\bigotimes_{J_h \in \pi_J} \left(\bigotimes_{J_h \supseteq J' \in \pi'_J} \begin{pmatrix} (diag) \\ s_{J'}^{-1} \end{pmatrix} \right) \circ s_{J_h} \right) \right) \\ &= \bigotimes_{H_l \in \pi_{I \cup J}} \left(\bigotimes_{H_l \supseteq K \in \pi'_{I \cup J}} \begin{pmatrix} (diag) \\ s_K^{-1} \end{pmatrix} \right) \circ s_{H_l} = z_{I \cup J;\pi_{I \cup J},\pi'_{I \cup J}} \end{aligned}$$

which proves (54). \square

Theorem 1 (i) The family of C^* -algebras defined by (49) is factorizable.

(ii) The operators defined by (50) are factorizable.

Proof. We apply Definition 8 to the case in which the family \mathcal{F} is the family of bounded Borel sets in \mathbb{R} . By induction it will be sufficient to prove that,

if I, J are disjoint bounded Borel sets in \mathbb{R} , then there exists a C^* -algebra isomorphism

$$u_{I,J} : \mathcal{W}_{1,n;I} \otimes \mathcal{W}_{1,n;J} \rightarrow \mathcal{W}_{1,n;I \cup J}$$

Since $\mathcal{W}_{1,n;I} \otimes \mathcal{W}_{1,n;J}$ is the inductive limit of the system (53) and $\mathcal{W}_{1,n;I \cup J}$ is the inductive limit of the system (52), the statement follows from Lemma 6 because isomorphic inductive systems have isomorphic inductive limits.

The factorizability of the operators (50) follows from the identity (49). \square

From Theorem 1 it follows that, if $I \subset J$ are bounded Borel sets in \mathbb{R} , then the map

$$j_{I;J} : w_I \in \mathcal{W}_{1,n;I} \rightarrow w_I \otimes 1_{J \setminus I} \in \mathcal{W}_{1,n;J} \quad (56)$$

is a C^* -algebra isomorphism. Since clearly, for $I \subset J \subset K$ bounded Borel sets in \mathbb{R} , $1_{J \setminus I} \otimes 1_{K \setminus J} \equiv 1_{K \setminus I}$, it follows that

$$\{(\mathcal{W}_{1,n;I}), (j_{I;J}), I \subset J \in \text{bounded Borel sets in } \mathbb{R}\} \quad (57)$$

is an inductive system of C^* -algebras.

Definition 9 *The inductive limit of the system (57) will be denoted*

$$\{\mathcal{W}_{1,n;\mathbb{R}}, (j_I), I \in \text{bounded Borel sets in } \mathbb{R}\}$$

Since the $j_I : \mathcal{W}_{1,n;I} \rightarrow \mathcal{W}_{1,n;\mathbb{R}}$ are injective embeddings, the family $(j_I(\mathcal{W}_{1,n;I}))$ is factorizable and one can introduce the more intuitive notation:

$$j_I(\mathcal{W}_{1,n;I}) \equiv \mathcal{W}_{1,n;I} \otimes 1_{I^c}$$

9 Existence of factorizable states on $\mathcal{W}_{1,n;\mathbb{R}}$

In the notation (38) and with the operators $W_I(u, P)$ defined by (50), using factorizability of the family $(\mathcal{W}_{1,n;I})$ and of the corresponding generators, for any $I \subset \mathbb{R}$ bounded Borel and any finite partition π of I , we will use the identifications

$$\begin{aligned} \mathcal{W}_{1,n;I} &\equiv j_I(\mathcal{W}_{1,n;I}) \equiv \mathcal{W}_{1,n;I} \otimes 1_{I^c} \subset \mathcal{W}_{1,n;\mathbb{R}} \\ \mathcal{W}_I(u, P) &\equiv \bigotimes_{I_0 \in \pi} \mathcal{W}_{I_0}(u, P) \quad ; \quad \forall (u, P) \in \mathbb{R} \times \mathbb{R}_n[X] \\ W_I(u, P) &\equiv \bigotimes_{I_0 \in \pi} W_{I_0}(u, P) \quad ; \quad \forall (u, P) \in \mathbb{R} \times \mathbb{R}_n[X] \end{aligned} \quad (58)$$

omitting from the notations the isomorphisms implementing these identifications.

Definition 10 *A state φ on $\mathcal{W}_{1,n;\mathbb{R}}$ is called factorizable if for every $I \subset \mathbb{R}$ bounded Borel, for every finite partition $\pi = (I_j)_{j \in F}$ of I and for every $W_I(u, P)$ as in (58), one has:*

$$\varphi(W_I(u, P)) = \prod_{j \in F} \varphi(W_{I_j}(u, P)) \quad ; \quad \forall (u, P) \in \mathbb{R} \times \mathbb{R}_n[X] \quad (59)$$

The map (14) can be used to lift the Fock state φ_F on $\mathcal{W}_{F,1,n}$ to a state, denoted φ_0 , on $\mathcal{W}_{1,n}^0$ through the prescription

$$\varphi_0(W^0(u, P)) := \varphi_F(W(u, P)) \quad (60)$$

($W^0(u, P) \in \mathcal{W}_{1,n}^0$, $W(u, P) \in \text{Un}(\Gamma(\mathbb{C}))$). Then, using the maps \tilde{z}_I defined by (50), for each bounded Borel set $I \subset \mathbb{R}$, one can define the state φ_I on $\tilde{z}_I(\mathcal{W}_{1,n;I}^0) \subset \mathcal{W}_{1,n;I}$ through the prescription that, for each $W_I^0(u, P) \in \mathcal{W}_{1,n;I}^0$, one has

$$\varphi_I(W_I(u, P)) = \varphi_I(\tilde{z}_I(W_I^0(u, P))) := \varphi_F(W(\hat{k}_I(u, P))) \quad (61)$$

Theorem 2 Under the assumption (29), if $n = 1$ then there exists a factorizable state φ on $\mathcal{W}_{1,n;\mathbb{R}}$ such that, for each bounded Borel set $I \subset \mathbb{R}$, one has

$$\varphi(W_I(u, P)) = \varphi_F(W(\hat{k}_I(u, P))) \quad ; \quad \forall (u, P) \in \mathbb{R} \times \mathbb{R}_n[X] \quad (62)$$

If $n \geq 2$, no such state exists.

Proof. Let I be a fixed bounded Borel set in \mathbb{R} and let π be a finite partition of I . From Definition 10 we know that φ is factorizable if and only if for every $I \subset \mathbb{R}$ bounded Borel set, for every finite partition π of I and for every $W_I(u, P)$ as in (58), (59) holds. If condition (62) is satisfied, the identity (59) becomes equivalent to:

$$\varphi_F(W(\hat{k}_I(u, P))) = \prod_{I_j \in \pi} \varphi_F(W(\hat{k}_{I_j}(u, P))) \quad ; \quad \forall (u, P) \in \mathbb{R} \times \mathbb{R}_n[X] \quad (63)$$

Thus the statement of the theorem is equivalent to say that, for $n = 1$ the identity (63) is satisfied and, for $n \geq 2$, not.

- **Case** $n = 1$. For $P = a_0 + a_1X$ and $u \in \mathbb{R}$, recalling the definition (35) of \hat{k}_I , one knows that

$$W(\hat{k}_I(u, P)) = e^{i(u|I|^{\frac{1}{2}}p + a_0|I|1 + a_1|I|^{\frac{1}{2}}q)}$$

whose Fock expectation is known to be

$$\varphi_F(W(\hat{k}_I(u, P))) = e^{-|I|(u^2 + a_1^2)/4} e^{ia_0|I|} = (\varphi_F(W(u, P)))^{|I|} \quad (64)$$

It follows that

$$\prod_{I_j \in \pi} \varphi_F(W(\hat{k}_{I_j}(u, P))) = \prod_{I_j \in \pi} (\varphi_F(W(u, P)))^{|I_j|}$$

Therefore, if $a_I = 1$, then

$$\varphi_F(W(\hat{k}_I(u, P))) = (\varphi_F(W(u, P)))^{|I|} = \varphi_F(W(u, P))$$

- **Case** $n \geq 2$. Since, for $n \geq 2$, the 1-mode n -th degree Heisenberg \ast -Lie algebra $heis_{\mathbb{R}}(1, n)$ contains a copy of $heis(1, 2)$ (see Definition 1), the algebra $\mathcal{W}_{1,n;\mathbb{R}}$ contains a copy of $\mathcal{W}_{1,2;\mathbb{R}}$. Therefore the non existence of a factorizable state on $\mathcal{W}_{1,2;\mathbb{R}}$, satisfying (62), will imply the same conclusion for $\mathcal{W}_{1,n;\mathbb{R}}$. In the case $n = 2$, let $P = a_0 + a_1X + a_2X^2$ and $u \in \mathbb{R}$. Then, using again $a_I = 1$, (7) and (30) one has

$$W(\hat{k}_I(u, P)) = e^{i(|I|^{\frac{1}{2}}up + a_0|I|1 + a_1|I|^{\frac{1}{2}}q + a_2q^2)}$$

and from [10] (Theorem 2), one knows that

$$\begin{aligned} \varphi_F(W(\hat{k}_I(u, P))) &= (1 - 2iA)^{-\frac{1}{2}} e^{ia_0|I|} e^{\frac{4C^2(A^2 + 2iA) - 3|M|^2}{6(1 - 2iA)}|I|} \\ &= (1 - 2iA)^{-\frac{1}{2}} \left(e^{ia_0} e^{\frac{4C^2(A^2 + 2iA) - 3|M|^2}{6(1 - 2iA)}} \right)^{|I|} \end{aligned}$$

where $A = \frac{a_2}{\sqrt{2}}$, $B = \frac{a_1}{\sqrt{2}}$, $C = \frac{u}{\sqrt{2}}$ and $M = B + iC$. On the other hand, if $\pi \in Part_{fin}(I)$ with $|\pi| > 1$, then

$$\begin{aligned} &\prod_{I_j \in \pi} \varphi_F(W(\hat{k}_{I_j}(u, P))) \\ &= \prod_{I_j \in \pi} \left((1 - 2iA)^{-\frac{1}{2}} \left(e^{ia_0} e^{\frac{4C^2(A^2 + 2iA) - 3|M|^2}{6(1 - 2iA)}} \right)^{|I_j|} \right) \\ &= (1 - 2iA)^{-\frac{|\pi|}{2}} \left(e^{ia_0} e^{\frac{4C^2(A^2 + 2iA) - 3|M|^2}{6(1 - 2iA)}} \right)^{|I|} \\ &\neq (1 - 2iA)^{-\frac{1}{2}} \left(e^{ia_0} e^{\frac{4C^2(A^2 + 2iA) - 3|M|^2}{6(1 - 2iA)}} \right)^{|I|} = \varphi_F(W(\hat{k}_I(u, P))) \end{aligned}$$

□

Lemma 7 In the case $n = 1$, the choice of the isomorphism \hat{s}_I (see Lemma 3) given by

$$\begin{aligned}\hat{s}_I(L_0((\chi_I))) &= a_I i |I| 1 \\ \hat{s}_I(L_2(\chi_I)) &= a_I i |I|^{\frac{1}{2}} p \\ \hat{s}_I(L_1(\chi_I)) &= a_I i |I|^{\frac{1}{2}} q\end{aligned}$$

gives rise to a factorizable state satisfying (62) if and only if the map $I \subset \mathbb{R} \mapsto a_I$ has the form

$$a_I := \frac{1}{|I|} \int_I p(s) ds$$

for all Borel subsets $I \subseteq \mathbb{R}$ where $p(\cdot)$ is a locally integrable almost everywhere strictly positive function on \mathbb{R} . In this case the factorizable state will be translation invariant if and only if $p(\cdot)$ is a strictly positive constant.

Proof. In the case $n = 1$, if $a_I \neq 1$, then the expression for $W(\hat{k}_I(u, P))$ becomes

$$W(\hat{k}_I(u, P)) = e^{i(u|I|^{\frac{1}{2}} a_I^{1/2} p + a_0 |I| a_I 1 + a_1 |I|^{\frac{1}{2}} a_I^{1/2} q)}$$

consequently its Fock expectation is

$$\varphi_F(W(\hat{k}_I(u, P))) = e^{-|I| a_I (u^2 + a_1^2)/4} e^{i a_0 a_I |I|} = (\varphi_F(W(u, P)))^{a_I |I|} \quad (65)$$

Therefore the factorizability condition (63) can hold if and only if the map $I \subset \mathbb{R} \mapsto a_I |I|$ is a finitely additive measure. In this case, by construction it will be absolutely continuous with respect to the Lebesgue measure hence there will exist a locally integrable almost everywhere positive function $p(\cdot)$ satisfying

$$a_I |I| := \int_I p(s) ds \quad ; \quad \forall \text{ Borel } I \subseteq \mathbb{R}$$

$p(\cdot)$ must be almost everywhere strictly positive because, by Lemma 3, $a_I > 0$ for any Borel set $I \subseteq \mathbb{R}$. This proves the first statement of the lemma. The second one follows because the Lebesgue measure is the unique translation invariant positive measure on \mathbb{R} . □

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